5.6 phase plane analysis

Monday, March 22, 2021 12:06 PM

Consider a 2D autonomous system

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y) \quad \text{where} \quad \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \text{ are continuous.}$$

Then by existence/uniqueness, only one solution passes through each pt in space, So we can draw out vector fields in the phase plane.

Option 1: Draw out vector fields fields for each pt using $\left(\frac{dx}{dt}, \frac{dy}{dt}\right)$

Option 2: Use nullcline analysis.

a set where $f(x,y) = C$ a constant.

Def. 5.7 The x-zero isocline (or x-nullcline) is the set of all points (x,y) satisfying $f(x,y)=0$.
The y-zero isocline (or y-nullcline). is the set of all points (x,y) satisfying $g(x,y)=0$.

Note: The x- and y-nullclines intersect at equilibria.
Also, on an x-nullcline, solutions can only move up/down
y-nullcline, solutions can only move left/right.

Ex. 5.13

$$\frac{dx}{dt} = xy - y = y(x-1) = f(x,y)$$

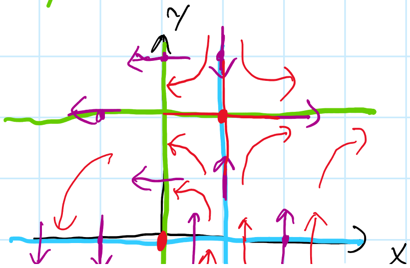
$$\frac{dy}{dt} = 2x - xy = x(2-y) = g(x,y)$$

$f(0,3) = -3$

$g(0,3) = 0$

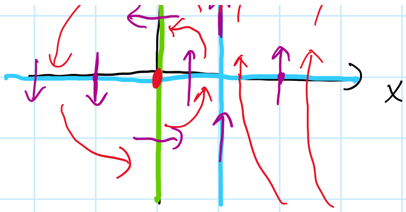
x-nullcline: $0 = y(x-1) \Rightarrow y=0$ or $x=1$

y-nullcline: $0 = x(2-y) \Rightarrow y=2$ or $x=0$



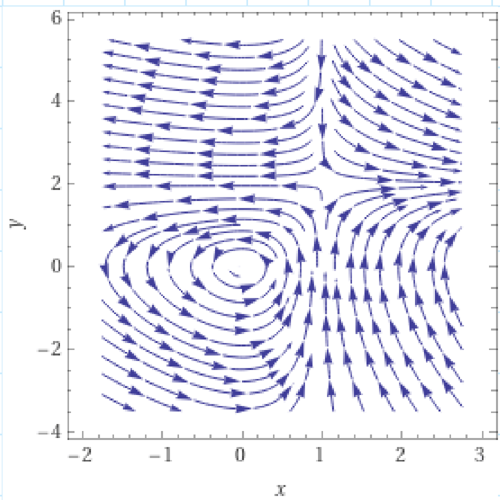
Equilibria: $(0,0)$, $(1,2)$

Exercise for viewer: Arrow direction along a nullcline



Exercise for viewer: Arrow direction along a nullcline varies continuously except at an equilibrium, where it'll change direction if $\det(J) \neq 0$.

$$J(x,y) = \begin{pmatrix} y & x-1 \\ 2-y & -x \end{pmatrix}$$



$$J(0,0) = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$$

$\det = 2$, $Tr = 0$

$$\lambda_{1,2} = \underbrace{\pm i\sqrt{2}}$$

$$J(1,2) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$$

$\det = -2$ $\quad Tr = 1$

$\lambda_1 = 2 \qquad \lambda_2 = -1$

$$\underbrace{v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\text{unstable manifold}} \qquad \underbrace{v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{stable manifold}}$$